5 Functions

Basic Definition of Functions

- In mathematics, a function is usually referred to as a mapping $f: A \to B$ that maps each element in domain A to a new element in another domain B. Essentially, a function can be defined as a relation having specific properties.

Definition: Functions

A relation $f \subseteq A \times B$ is said to be a **function (函数)** from A to B if

① $(\forall x)(\forall y)(\forall y')((xfy \wedge xfy') \to y = y')$

② $(\forall x)(x \in A \to (\exists y)(y \in B \wedge xfy))$

- We make the following remarks regarding the above definition of function

1. The first condition says that for each element $x \in A$, it is related to a unique element in B. We usually denote by $f(x)$ the unique element such that $\langle x, f(x)\rangle \in f$.
2. The second condition says that any element in A is related to some element in B, i.e., $\mathsf{dom}(f) = A$. Therefore, a function is also called a total function (全函数). If $\mathsf{dom}(f) \subset A$, i.e., $f(x)$ is undefined for some $x \in A$, then we call f a partial function (部分函数). For example, $f = \{\langle 1,2\rangle, \langle 2,3\rangle\}$ is a function from $\{1,2\}$ to $\{2,3\}$, but is just a partial function from $\{1,2,3\}$ to $\{1,2,3\}$.
3. Finally, we note that f is a function from A to B does not imply that f^{-1} is a function from B to A. For example $f = \{\langle 1,3\rangle, \langle 2,3\rangle, \langle 3,1\rangle\}$ is a function, but $f^{-1} = \{\langle 3,1\rangle, \langle 3,2\rangle, \langle 1,3\rangle\}$ is not a function.
4. Sometimes, it is not convenient to list all pairs in f. It we can write $f(x)$ in a close-form, then usually we use $\boxed{f: A \to B, f: x \mapsto y}$ to denote that f is a function from A to B and $\langle x, y\rangle \in f$. For example, $f: \mathbb{R} \to \mathbb{R}, f: x \mapsto 2x + 1$ means that $f = \{\langle x, y\rangle : y = 2x + 1\} \subseteq \mathbb{R} \times \mathbb{R}$.
5. The set of all functions from A to B is denoted by $A_B = \{f : (f : A \to B)\}$. We note the following differences:
   - Since $\emptyset : \emptyset \to B$, we have $\emptyset_B = \{\emptyset\}$ for any B
   - $f : A \to \emptyset$ does not exist when $A \neq \emptyset$. Therefore, $A_\emptyset = \emptyset$ when $A \neq \emptyset$.

Example: Commonly Used Functions

- Constant Function (常函数): $f: A \to B$ such that $(\forall x)(f(x) = c)$ for some $c \in B$.
- Identity Function (恒等函数): $I_A : A \to A$ such that $(\forall x)(f(x) = x)$
- Indicator Function (特征函数): Let $A \subseteq E$. Then $\mathcal{X}_A : E \to \{0,1\}$ is defined by $\mathcal{X}_A(x) = 1$ iff $x \in A$.

Injections, Surjections & Bijections

- Depending on how a function maps each element from the domain to its range, e.g., one-to-one or not, we can classify functions as follows:

Definition: Classification of Functions

Let $f : A \to B$ be a function. Then we say f is a

– **injection (单射)**: if $(\forall x_1 \in A)(\forall x_2 \in A)(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$

– **surjection (满射)**: if $\mathsf{ran}(f) = B$

– **bijection (双射)**: if it is both an injection and a surjection.

Injection is also called one-to-one, surjection is also called onto and bijection is also called one-to-one correspondence.

Examples

– For $A = \{1, 2, 3\}, B = \{1, 2, 3, 4\}$, $f_1 = \{\langle 1, 2\rangle, \langle 3, 2\rangle, \langle 2, 3\rangle\}$ is neither an injection nor a surjection, but $f_2 = \{\langle 1, 2\rangle, \langle 3, 4\rangle, \langle 2, 1\rangle\}$ is an injection.

– For $f : \mathbb{N} \to \mathbb{N}, f : x \mapsto 2x$, it is an injection but is NOT a surjection. However, for $f : \mathbb{R} \to \mathbb{R}, f : x \mapsto 2x$, it is a bijection.

– $\emptyset : \emptyset \to B$ is always an injection. Furthermore, it is a bijection when $B = \emptyset$.

- Since a function $f \subseteq A \times B$ is also a relation, it makes sense to talk about its converse $f^{-1} \subseteq B \times A$. However, f^{-1} may not be a function because either (i) $\mathsf{dom}(f^{-1}) \subset B$; or (ii) $\langle b, a_1\rangle, \langle b, a_2\rangle \in f^{-1}$ for different a_1 and a_2. Therefore, only when f is a bijection, we have that $f^{-1} : B \to A$ is a function, and we call f^{-1} the **inverse function (逆函数)** of f. In fact f^{-1} is still a bijection such that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

- Also, for $g : A \to B$ and $f : B \to C$, it is easy to check that $f \circ g : A \to C$ is also a function such that for any $x \in A$, we have $f \circ g(x) = f(g(x))$. Furthermore, we have

Theorem: Composition of Functions

Let $g : A \to B$ and $f : B \to C$. Then

① If f and g are bijections (respectively, injections or surjections), then $f \circ g$ is also a bijection (respectively, injection or surjection).

② If $f \circ g$ is a surjection, then f is a surjection.

③ If $f \circ g$ is an injection, then g is an injection.

Note that when $f \circ g$ is a surjection, g may not be a surjection. For example, for $A = \{1, 2\}, B = \{3, 4\}, C = \{5\}$, consider $g = \{\langle 1, 3\rangle, \langle 2, 3\rangle\}$ and $f = \{\langle 3, 5\rangle, \langle 4, 5\rangle\}$. Then $f \circ g = \{\langle 1, 5\rangle, \langle 2, 5\rangle\}$ is surjection from A to C but g is not surjection from A to B.

Cardinality and Equinumerous

- Given a set A, we denote by $|A|$ or $\mathsf{card}(A)$ the cardinality (基数) of A, which is the number of all elements in it. It is easy to compare cardinalities for finite sets, e.g., $|\{1,2,3\}| = 3 > |\{a,b\}| = 2$. However, the question is how to compare infinite sets? For example, for $\mathbb{N}$ and $\mathbb{N}_{even}$, which one has "more" elements? Our basic idea is to use bijection to establish equivalence of elements numbers.

Definition: Equinumerous

Let A and B be two sets. We say A and B are **equinumerous (等势)**, denoted by $A \approx B$, if there exists a bijection from A to B.

- Using the concept of equinumerous, we can classify the size of a set as follows.

Definition: Finite Sets & Infinite Sets

Let A be a set. Then we say A is

- **finite (有限的)**: if $A \approx \{1, 2, \cdots, n\}$ for some n and we define $|A| = n$.
- **infinite (无限的)**: if it is not finite.

Furthermore, for infinite set A, we say A is **countably infinite (可数无限的)** if $A \approx \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of all natural numbers. The cardinality of infinite set is defined as follows:

- $|\mathbb{N}| = \aleph_0$, where $\aleph_0$ is called the **Aleph Zero (阿列夫零)**;
- If $|A_k| = \aleph_k$, then $|2^{A_k}| = \aleph_{k+1}$, where $\aleph_k$ is called the Aleph k (阿列夫 k).

- We use bijection to define that two sets have the same cardinality. Similarly, we say A is **dominated (支配)** by B, denoted as $A \preceq B$, if there exists an injection $f : A \to B$. We write $A \prec B$ if $A \preceq B$ and $A \not\approx B$. Intuitively, $A \prec B$ means that A has "less" elements than B. It is easy to guess that $A \approx B$ iff $(A \preceq B) \wedge (B \preceq A)$. In fact, it is correct, but its proof is actually much difficult than we may guess.

Theorem: Schröder-Berstein's Theorem

If $A \preceq B$ and $B \preceq A$, then $A \approx B$.

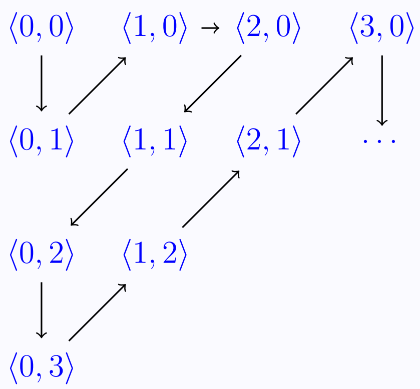
- For finite sets, it is easy to compare their sizes. Clearly, if $A \subset B$, then we have $|A| < |B|$. For example, we have $\{1, 2, 3, 4\} \approx \{a, b, c, d\} \approx \{$离, 散, 数, 学$\} \not\approx \{$数, 学分$\}$. However, for infinite sets, $A \subset B$ does not imply that $|A| < |B|$.
  - $\mathbb{N}_{even} \approx \mathbb{N}$ since we have bijection $f : x \mapsto 2x$.
  - $(0, 1) \approx \mathbb{R}$ since we have bijection $f(x) = \tan(\pi(x - \frac{1}{2}))$
  - $\mathbb{R} \approx \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ since we have bijection $f : x \mapsto e^x$

From Countable Infinity to Uncountable Infinity

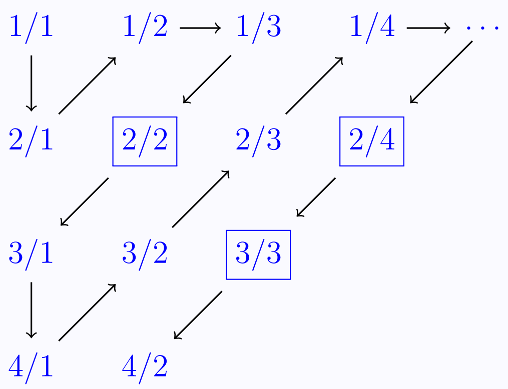
- Note that any countably infinite set A is equinumerous to natural numbers $\mathbb{N}$. To establish a bijection from A to $\mathbb{N}$, essentially, it asks to find a way to list of elements of A in order. Using this fact, we can show that the Cartisan product of natural numbers $\mathbb{N} \times \mathbb{N}$ and rational numbers $\mathbb{Q}$ are all countably infinite.

  Result: $\boxed{\mathbb{N} \approx \mathbb{N} \times \mathbb{N} \approx \mathbb{Q}}$

  To prove the above facts, we just need to list all elements in $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q}$ as follows:



$\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$



$\mathbb{Q} \approx \mathbb{N}$

- However, we cannot list all real numbers in order, which tells that $\mathbb{R}$ is uncountable.

  Result: $\boxed{\mathbb{N} \not\approx \mathbb{R}}$

  The proof of the above fact is the famous Cantor's diagonal argument. Note that $(0, 1) \approx \mathbb{R}$; it suffices to show $(0, 1) \not\approx \mathbb{R}$. Specifically, we assume that, by some means, all real numbers in $(0, 1)$ can be listed in order. Suppose that it is listed such that the ith real number is $0.a_i b_i c_i d_i e_i \ldots$. Then we consider a new real number $0.abcde\ldots$ such that $a \neq a_1, b \neq b_2, c \neq c_3, d \neq d_4, \ldots$. This means that this new number is not listed! Therefore, there is no way to list all elements in $(0, 1)$, which means that it is uncountable.

- The above result shows that $\mathbb{R}$ is "larger" than $\mathbb{N}$. The following result shows that we can always find a set larger than a given set

  Result: $\boxed{\text{for any set } A, \text{ we have } A \not\approx 2^A}$

  We prove the above result by contradiction. Assume that there exists a bijection $f : A \to 2^A$. We define set $Y = \{x \in A : x \notin f(x)\}$. Clearly, $Y \in 2^A$. Since f is a bijection, we can choose the unique element $y \in A$ such that $f(y) = Y$. However, there is a problem

  - if $y \in Y$, then $y \notin f(y) = Y$;
  - if $y \notin Y$, then $y \in f(y) = Y$.

- We have shown using the Cantor's diagonal argument that $\mathbb{N} \not\approx \mathbb{R}$. In fact, using the Schröder-Berstein's theorem, we can further argue that $|\mathbb{R}| = \aleph_1$, i.e.,

  Result: $\boxed{2^{\mathbb{N}} \approx \mathbb{R}}$