4 Relations

Basic Definition of Relations

- In our daily life, a **relation (关系)** is usually referred to as some kind of connections between things. For example, "Father" is a relation and if Alice is the father of Bob, then we say that they have the "Father" relation. Note that Alice is the father of Bob does not means that Bob is the father of Alice. Therefore, we note that "relation" is something having direction and we need to use ordered pair $\langle \mathsf{Alice}, \mathsf{Bob} \rangle$ to connect them. Of course, if we have a group of people, then there may have multiple pairs of two persons having the "Father" relation. Therefore, we define a relation as a set of ordered pairs as follows.

Definition: Relation

A **binary relation (二元关系)** from a set A to a set B is a subset of $A \times B$. Hence, a relation R consists of ordered pairs $\langle a, b \rangle$, where $a \in A$ and $y \in B$. If $\langle a, b \rangle \in R$, we say that a is related to b under relation R, and we also write aRb .

- There are many different ways to represent a relation. The most direct approach is to list all elements (ordered pairs) in the relation. For example, for sets $A = \{a, b, c, d\}$ and $B = \{0, 1\}$, we can define relation $R \subseteq A \times B$ explicitly by

$$R = \{\langle a, 0 \rangle, \langle b, 1 \rangle, \langle c, 0 \rangle, \langle d, 1 \rangle\}$$

Essentially, the above relation R is a Boolean function that assigns each elements in A a value 0 or 1. However, in general, it is possible that both $\langle a, 0 \rangle, \langle a, 1 \rangle \in R$. In this case, it is not a function because we cannot assign an element two different values.

- Essentially, a relation puts all related pairs of elements $\langle a, b \rangle$ together in a set. Therefore, it is a subset of $A \times B$. Depending on what do you mean by "related", the generic notation R can be replaced by specific symbols for the sake of understanding.

Example

Let $A = \{1, 2, 3\}$. The following relation $R_1 \subseteq A \times A$ is essentially the "smaller than or equal to" relation on A

$$R = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle\} \subseteq A \times A$$

If we write relation R as symbol "$\leq$", then $1 \leq 3$ means that $\langle 1, 3 \rangle \in \leq$, which is read as "1 is related to 3 under "$\leq$" relation" (or $\leq$-related). Of course, we usually read is as "1 is smaller than or equal to 3" directly.

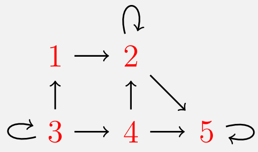
- Usually, we investigate binary relations. In general, we can extend it to an n-ary relation from A_1 to A_n, which is a subset $R \subseteq A_1 \times A_2 \times \cdots \times A_n$.

More Concepts about Relations

- An intuitive way for representing relation $R \subseteq A \times B$ is to use the **relation graph (关系图)**. Specifically, we draw each element in $A \cup B$ as a node and draw an arrow $x \to y$ if $\langle x, y\rangle \in R$. If $R \subseteq A \times A$ is a relation on A itself, then we just need to draw elements in A.

Example: Relation Graph

Let $A = \{1, 2, 3, 4, 5\}$ and $R = \{\langle 1, 2\rangle, \langle 2, 2\rangle, \langle 2, 5\rangle, \langle 3, 1\rangle, \langle 3, 3\rangle, \langle 3, 4\rangle, \langle 4, 2\rangle, \langle 4, 5\rangle, \langle 5, 5\rangle\} \subseteq A \times A$. The relation graph of R, denoted by $G(R)$, is shown on the right.



- If $R \subseteq A \times A$ is a relation on A, then we define the following special relations:
  - **Identity Relation (恒等关系)**: $I_A = \{\langle x, x\rangle : x \in A\} \subseteq A \times A$
  - **Universal Relation (全域关系)**: $E_A = \{\langle x, y\rangle : x \in A \wedge y \in A\} = A \times A$
  - **Empty Relation (空关系)**: $\emptyset$

  Under identity relation, each element $x \in A$ is related (and only related) to itself. Therefore, when we say two elements $x, y \in A$ are "precisely the same", which is usually written as $x = y$, essentially, we mean that $\langle x, y\rangle \in I_A$.

- Let $R \subseteq A \times B$ be a relation from A to B. We also define
  - the **Domain (定义域)** of R: $\mathsf{dom}(R) = \{x : (\exists y)(\langle x, y\rangle \in R)\}$
  - the **Range/Image (值域)** of R: $\mathsf{ran}(R) = \{y : (\exists x)(\langle x, y\rangle \in R)\}$
  - the **Field (域)** of R: $\mathsf{fld}(R) = \mathsf{dom}(R) \cup \mathsf{ran}(R)$

  For example, for sets $A = \{a, b, c\}, B = \{b, c, d\}$ and relation $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle b, d\rangle\} \subseteq A \times B$, we have $\mathsf{dom}(R) = \{a, b\}, \mathsf{ran}(R) = \{b, c, d\}$ and $\mathsf{fld}(R) = \{a, b, c, d\}$.

- Essentially, $\mathsf{dom}(R)$ is the set of elements in A that are related to some elements in B, and $\mathsf{ran}(R)$ is the set of elements in B, for which some elements in A are related to them. Then $\mathsf{fld}(R)$ is the set of elements that are either in the range or in the domain of R. Recall that, in terms of set, each order pair is actually $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$. Then we know that, for any relation $R \subseteq A \times B$, we have

$$\mathsf{fld}(R) = \bigcup\bigcup R$$

Proof: For inclusion "$\subseteq$", $x \in \mathsf{fld}(R)$ means $x \in \mathsf{dom}(R)$ or $x \in \mathsf{ran}(R)$. Therefore, $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$ for some y, i.e., either $\{\{x\}, \{x, y\}\} \in R$ or $\{\{y\}, \{x, y\}\} \in R$. For both cases, we have $\{x, y\} \in \bigcup R$, which implies that and $x \in \bigcup\bigcup R$.

For inclusion "$\supseteq$", for any $t \in \bigcup\bigcup R$, we have $\{\{t\}, \{t, u\}\} \in R$ or $\{\{u\}, \{u, t\}\} \in R$ for some u. This means that either $t \in \mathsf{dom}(R)$ or $t \in \mathsf{ran}(R)$, which means $t \in \mathsf{fld}(R)$.

Relation Operations

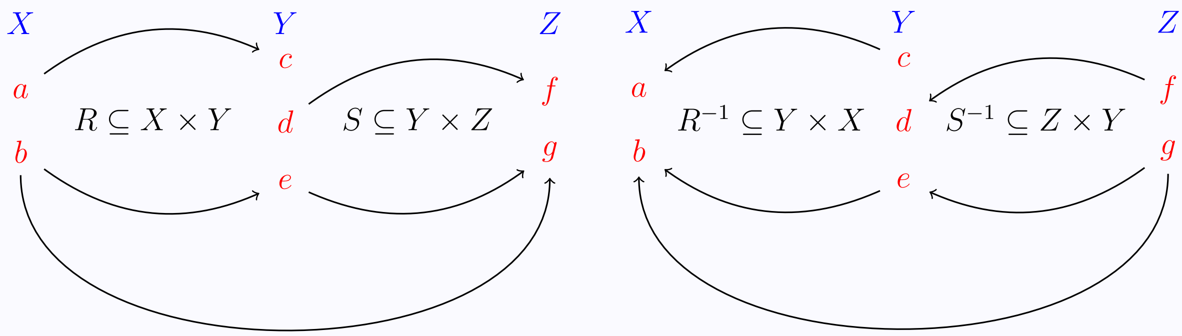
- First, we define the following relation operations that we investigate.

Definition: Relation Operations

Let $R \subseteq X \times Y$ be a relation from X to Y, $S \subseteq Y \times Z$ be a relation from Y to Z and A be a new set. Then we define the followings:

- the **converse (逆)** of R is a new relation $R^{-1} \subseteq Y \times X$ s.t.
$$R^{-1} = \{\langle x, y\rangle : \langle y, x\rangle \in R\}$$
which simply swaps the direction of each ordered pair.
- the **restriction (限制)** of R to set A is a new relation $R \restriction A \subseteq A \times Y$ s.t.
$$R \restriction A = \{\langle x, y\rangle : \langle x, y\rangle \in R \wedge x \in A\}$$
which restricts our attention by only considering those ordered pairs whose first elements are in A. In fact, we have $R \restriction A = R \cap (A \times Y)$.
- the **image (象)** of R on set A is a set $R[A] \subseteq Y$ s.t.
$$R[A] = \{y : (\exists x)(x \in A \wedge \langle x, y\rangle \in R)\}$$
which is the set of elements in Y that are related by some elements in $A \cap X$.
- the **composite (合成)** of R and S is a new relation $S \circ R \subseteq X \times Z$ s.t.
$$S \circ R = \{\langle x, z\rangle : (\exists y)(\langle x, y\rangle \in R \wedge \langle y, z\rangle \in S)\}$$
which actually "connects" elements in X and Z by elements in Y. That is, if $\langle x, z\rangle \in S \circ R$, then they must be "connected" by some $y \in Y$ in the sense that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S$.

- **Example 1:** Let us consider three sets $X = \{a, b\}, Y = \{c, d, e\}, Z = \{f, g\}$, and two relations $R = \{\langle a, c\rangle, \langle b, e\rangle\} \subseteq X \times Y$, $S = \{\langle d, f\rangle, \langle e, g\rangle\} \subseteq Y \times Z$. Then we have $S \circ R = \{\langle b, g\rangle\}$ and $R^{-1} \circ S^{-1} = \{\langle g, b\rangle\}$ shown as follows.



- **Example 2:** Let $A = \{a, \{a\}, \{\{a\}\}\}$ and $R = \{\langle a, \{a\}\rangle, \langle \{a\}, \{\{a\}\}\rangle\} \subseteq A \times A$. Then

$R^{-1} = \{\langle \{a\}, a\rangle, \langle \{\{a\}\}, \{a\}\rangle\}$ $R \circ R = \{\langle a, \{\{a\}\}\rangle\}$ $R \restriction \{a\} = \{\langle a, \{a\}\rangle\}$

$R^{-1} \restriction \{a\} = \emptyset$ $R[\{a\}] = \{\{a\}\}$ $R[\{\{a\}\}] = \{\{\{a\}\}\}$

Properties of Relation Operations

- First we note that the converse of R swaps the domain and the range of R and the ordering of each element. We have the following results.

Theorem: Properties of Converse

Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$. Then

① $\mathsf{dom}\,(R^{-1}) = \mathsf{ran}(R)$ ② $\mathsf{ran}\,(R^{-1}) = \mathsf{dom}(R)$

③ $(R^{-1})^{-1} = R$ ④ $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Proof: ① and ② follows directly from the following equivalences:

$$x \in \mathsf{dom}(R^{-1}) \Leftrightarrow (\exists y)(\langle x, y\rangle \in R^{-1}) \Leftrightarrow (\exists y)(\langle y, x\rangle \in R) \Leftrightarrow x \in \mathsf{ran}(R)$$

For ③, we have $\langle x, y\rangle \in R \Leftrightarrow \langle y, x\rangle \in R^{-1} \Leftrightarrow \langle x, y\rangle \in (R^{-1})^{-1}$.

Now, let us prove ④. First, we note that $R^{-1} \subseteq Y \times X$ and $S^{-1} \subseteq Z \times Y$. Therefore, $R^{-1} \circ S^{-1} \subseteq Z \times X$, which matches the domain of $(S \circ R)^{-1} \subseteq Z \times X$ since $S \circ R \subseteq X \times Z$

$$\begin{aligned}\langle x, y\rangle \in (S \circ R)^{-1} \Leftrightarrow& \langle y, x\rangle \in S \circ R \Leftrightarrow (\exists z)(\langle y, z\rangle \in R \wedge \langle z, x\rangle \in S)\\ \Leftrightarrow& (\exists z)(\langle z, y\rangle \in R^{-1} \wedge \langle x, z\rangle \in S^{-1}) \Leftrightarrow \langle x, y\rangle \in R^{-1} \circ S^{-1}\end{aligned}$$

- Also, we note the associative law holds for composite as follows:

Theorem: Associative Law of Composite

Let $Q \subseteq X \times Y, S \subseteq Y \times Z$ and $R \subseteq Z \times W$. Then $(R \circ S) \circ Q = R \circ (S \circ Q)$.

Proof: The correctness follows from the following equivalences:

$$\begin{aligned}&\langle x, y\rangle \in (R \circ S) \circ Q\\ \Leftrightarrow& (\exists u)(\langle x, u\rangle \in Q \wedge \langle u, y\rangle \in R \circ S) \Leftrightarrow (\exists u)(\langle x, u\rangle \in Q \wedge (\exists v)(\langle u, v\rangle \in S \wedge \langle v, y\rangle \in R))\\ \Leftrightarrow& (\exists u)(\exists v)(\langle x, u\rangle \in Q \wedge \langle u, v\rangle \in S \wedge \langle v, y\rangle \in R) \Leftrightarrow (\exists v)(\exists u)(\langle x, u\rangle \in Q \wedge \langle u, v\rangle \in S \wedge \langle v, y\rangle \in R)\\ \Leftrightarrow& (\exists v)(\langle x, v\rangle \in S \circ Q \wedge \langle v, y\rangle \in R)\\ \Leftrightarrow& \langle x, y\rangle \in R \circ (S \circ Q)\end{aligned}$$

However, the communicative law does not hold for composition, i.e., $S \circ R \neq R \circ S$.

- Finally, we have the following distributive laws.

Theorem: Distributive Laws for Composite

① $R \circ (S \cup W) = (R \circ S) \cup (R \circ W)$ ② $(R \cup S) \circ W = (R \circ W) \cup (S \circ W)$

③ $R \circ (S \cap W) \subseteq (R \circ S) \cap (R \circ W)$ ④ $(R \cap S) \circ W \subseteq (R \circ W) \cap (S \circ W)$

We note that inclusion "$\supseteq$" does not hold for the last two. Try to come up counter-examples by yourself.

Classifications of Relations

- We note that many relations have some *inherent structural properties*. For example, for $\leq \subseteq \mathbb{R} \times \mathbb{R}$, if $x \leq y$ and $y \leq z$, we naturally have $x \leq z$. Essentially, it tells us that the "smaller than or equal to" relation $\leq$ is *transitive*. In the followings, we abstract some important properties and use them to classify relations.

Definition: Notions of Relations

Let $R \subseteq A \times A$ be a relation on A. We say that R is

– **reflexive (自反的)** if any element is related to itself, i.e.,

$$(\forall x)(x \in A \to xRx)$$

– **inreflexive (非自反的)** if no element is related to itself, i.e.,

$$(\forall x)(x \in A \to x\not{R}x)$$

– **symmetric (对称的)** if whenever x is related to y, y is related to x, i.e,

$$(\forall x)(\forall y)((x \in A \land y \in A \land xRy) \to yRx)$$

– **antisymmetric (反对称的)** if whenever x is related to y and $x \neq y$, y is not related to x, i.e,

$$(\forall x)(\forall y)\left((x \in A \land y \in A \land xRy \land yRx) \to x = y\right)$$

or equivalently

$$(\forall x)(\forall y)\left((x \in A \land y \in A \land xRy \land x \neq y) \to y\not{R}x\right)$$

– **transitive (传递的)** if whenever x is related to y and y is related to z, x is related to z, i.e.,

$$(\forall x)(\forall y)(\forall z)((x \in A \land y \in A \land z \in A \land xRy \land yRz) \to xRz)$$

- **Example 1**: Let us classify some relations in our daily life using the above concepts:
  – Subset relation is reflexive and transitive because any set is a subset of itself and $A \subseteq B \land B \subseteq C \Rightarrow A \subseteq C$.
  – Friend is usually as a symmetric relation because if I am your friend then you are my friend. It is not transitive because the friend of my friend may not be my friend.
  – Father should be an inreflexive relation because no one can be the father of himself. It is also anti-symmetric because two persons cannot be the father of each other.

- **Example 2**: Let $A = \{1, 2, 3\}$ be the set of elements. We have the following results:

Relation	Reflex.	Inreflex.	Sym.	AntiSym.	Trans.
I_A	✓	×	✓	✓	✓
E_A	✓	×	✓	×	✓
$R = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle\}$	×	×	×	✓	✓
$R = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 1,3\rangle\}$	×	✓	×	×	×

Properties of Relation Notions

- The following result shows that (anti-)symmetry can be checked easily.

Theorem

Let $R \subseteq A \times A$ be a relation on A. Then we have
① R is symmetric iff $R = R^{-1}$.
② R is antisymmetric iff $R \cap R^{-1} \subseteq I_A$.

The proofs of the above results are as follows:

① "⇒" Suppose that R is symmetric. Then $\langle x, y\rangle \in R$ iff $\langle y, x\rangle \in R$ iff $\langle x, y\rangle \in R^{-1}$.
"⇐" Suppose that $R = R^{-1}$. Then for any $\langle x, y\rangle \in R = R^{-1}$, we have $\langle y, x\rangle \in R$, which means that R is symmetric.

② "⇒" Suppose R is anti-symmetric. Then for any $\langle x, y\rangle \in R \cap R^{-1}$, we have $\langle x, y\rangle \in R \wedge \langle x, y\rangle \in R^{-1}$, which means that $\langle x, y\rangle \in R \wedge \langle y, x\rangle \in R$. However, R is anti-symmetric, which means that we can only have $x = y$. Therefore, $\langle x, y\rangle \in I_A$.
"⇐" Suppose $R \cap R^{-1} \subseteq I_A$. Then for any $x, y \in A$ such that $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$, we have $\langle x, y\rangle \in R \cap R^{-1} \subseteq I_A$. Therefore, we have $x = y$, which means that R is anti-symmetric.

- Similar to the above results, one can show (leave as homework) that
  - R is reflexive iff $I_A \subseteq R$
  - R is inreflexive iff $I_A \cap R = \emptyset$
  - R is transitive iff $R \circ R \subseteq R$.

- In mathematics, given some elements satisfying a desired property, if we take some operations and the resulting element also satisfies the property, then we say that this property is *closed* (封闭的) under this operation.

Theorem: Properties Closed under Operations

① If $R_1, R_2 \subseteq A \times A$ are reflexive, then $R_1^{-1}, R_1 \cap R_2, R_1 \cup R_2$ are reflexive.
② If $R_1, R_2 \subseteq A \times A$ are symmetric, then $R_1^{-1}, R_1 \cap R_2, R_1 \cup R_2$ are symmetric.
③ If $R_1, R_2 \subseteq A \times A$ are transitive, then $R_1^{-1}, R_1 \cap R_2$ are transitive.
④ If $R_1, R_2 \subseteq A \times A$ are antisymmetric, then $R_1^{-1}, R_1 \cap R_2$ are antisymmetric.

- The proofs of the above results are rather straightforward. One can verify them by excise. Here, we note that, transitivity and anti-symmetry are not closed under union. To see this, let us consider $R_1 = \{\langle 1, 2\rangle\}$ and $R_2 = \{\langle 2, 3\rangle\}$. Both of them are transitive, but $R_1 \cup R_2 = \{\langle 1, 2\rangle, \langle 2, 3\rangle\}$ is not because $\langle 1, 3\rangle \notin R_1 \cup R_2$. Also, consider $R_1 = \{\langle 1, 2\rangle\}, R_2 = \{\langle 2, 1\rangle\}$. Both of them are anti-symmetric but $R_1 \cup R_2 = \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$ is not.

Definitions of Closures

- Given a relation R, it may not satisfy some property. But in some case, we still need to contain all elements in R. Therefore, we have to extend R to a larger relation to satisfy the property and at the same time, add as less as possible.

Definition: Clousures

Let $R \subseteq A \times A$ be a relation on A. We say $R' \subseteq A \times A$ is the **reflexive closure (自反闭包)** of R if it is the smallest reflexive relation on A containing R, i.e.,

(i) R' is reflexive; and

(ii) $R \subseteq R'$: and

(iii) For any R'' satisfying (i) and (ii), we have $R' \subseteq R''$.

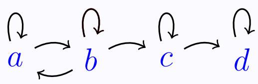
The **transitive closure (传递闭包)** and the **symmetric closure (对称闭包)** are defined analogously by replacing “reflexive” by “transitive” and “symmetric”, resp.

Notations: we use the following notations to denote different closures of R

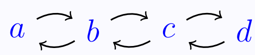
reflexive closure: $r(R)$, symmetric closure: $s(R)$, transitive closure: $t(R)$.

- **Example:** consider relation R, whose relation graph is 

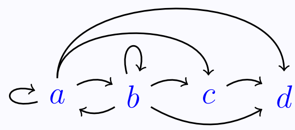
, we have



$r(R)$



$s(R)$



$t(R)$

- Following the definitions of closures, we have the following properties:

Theorem: Properties of Closures

① If R is symmetric/reflexive/transitive, then $s(R)=R$ or $r(R)=R$ or $t(R)=R$.

② If $R_1 \subseteq R_2$, then $r(R_1) \subseteq r(R_2), s(R_1) \subseteq s(R_2)$ and $t(R_1) \subseteq t(R_2)$

③ $r(R_1) \cup r(R_2) = r(R_1 \cup R_2)$ ④ $s(R_1) \cup s(R_2) = s(R_1 \cup R_2)$

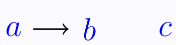
⑤ $t(R_1) \cup t(R_2) \subseteq t(R_1 \cup R_2)$

① and ② are straightforward. Now, let us prove ③; then ④ and ⑤ are similar.

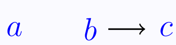
“$\subseteq$”: We observe that, since $R_1 \subseteq R_1 \cup R_2$, $r(R_1) \subseteq r(R_1 \cup R_2)$. Similarly, $r(R_2) \subseteq r(R_1 \cup R_2)$. Therefore, LHS$\subseteq$RHS.

“$\supseteq$”: First, we note that $R_1 \subseteq r(R_1) \wedge R_2 \subseteq r(R_2) \Rightarrow R_1 \cup R_2 \subseteq r(R_1) \cup r(R_2)$. Since $r(R_1)$ and $r(R_2)$ are reflexive, we know that $r(R_1) \cup r(R_2)$ is reflexive. Therefore, by the definition of reflexive clousure, we have $r(R_1 \cup R_2) \subseteq r(R_1) \cup r(R_2)$.

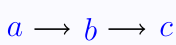
- Note that “$\supseteq$” of ⑤ above is not correct because transitivity is not preserved under union. Let us consider the following example.



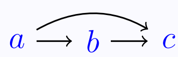
$R_1 = t(R_1)$



$R_2 = t(R_2)$



$t(R_1) \cup t(R_2)$



$t(R_1 \cup R_2)$

Construction of Reflexive and Symmetric Closures

- First, let us discuss how to construct reflexive closures. To make $R \subseteq A \times A$ reflexive, we need to make sure that all $\langle x, x\rangle$ are in R. This leads to the following result.

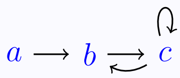
Theorem: Reflexive Closure

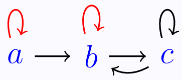
Let $R \subseteq A \times A$. Then $r(R) = R \cup I_A$.

Proof: We show that all three conditions in reflexive closure hold:

1. For any $x \in A$, we have $\langle x, x\rangle \in R \cup I_A$. So $R \cup I_A$ is reflexive.
2. Clearly, $R \subseteq R \cup I_A$.
3. To see "smallest", let us consider any R' such that R' is reflexive and $R \subseteq R'$. Then for any element $\langle x, y\rangle \in R \cup I_A$, we have either (i) $\langle x, y\rangle \in R$ or (ii) $\langle x, y\rangle \in I_A$.
   - If $\langle x, y\rangle \in R$, then $\langle x, y\rangle \in R'$.
   - If $\langle x, y\rangle \in I_A$, then $x = y$, which means that $\langle x, y\rangle \in R'$ since R' is reflexive.

   Therefore, we have $R \cup I_A \subseteq R'$, i.e., $R \cup I_A$ is the "smallest one".

- For example, for relation R : 

, we have $r(R)$: 

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- Second, let us discuss how to construct symmetric closures. To make $R \subseteq A \times A$ symmetric, we need to make sure that if $\langle x, y\rangle \in R$, then we should also add $\langle y, x\rangle$ to R. This leads to the following result.

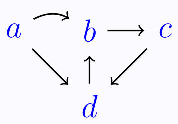
Theorem: Symmetric Closure

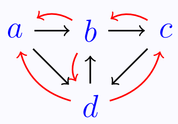
Let $R \subseteq A \times A$. Then $s(R) = R \cup R^{-1}$.

Proof: We show that all three conditions in reflexive closure hold:

1. It is symmetric $(R \cup R^{-1})^{-1} = R^{-1} \cup R = R \cup R^{-1}$.
2. Clearly, $R \subseteq R \cup R^{-1}$.
3. To see "smallest", let us consider any R' such that R' is symmetric and $R \subseteq R'$. Then $\langle x, y\rangle \in R \cup R^{-1}$ means that either (i) $\langle x, y\rangle \in R$ or (ii) $\langle x, y\rangle \in R^{-1}$.
   - If $\langle x, y\rangle \in R$, then we have $\langle x, y\rangle \in R'$.
   - If $\langle x, y\rangle \in R^{-1}$, then $\langle y, x\rangle \in R \subseteq R'$. However, since R' is symmetric, we have $\langle x, y\rangle \in R'$.

   Therefore, we have $R \cup R^{-1} \subseteq R'$, i.e., $R \cup R^{-1}$ is the "smallest one".

- For example, for relation R : 

, we have $s(R)$: 

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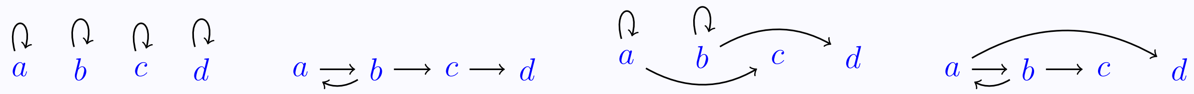
Construction of Transitive Closures

- The construction of transitive closures is bit more involved. First, for any $R \subseteq A \times A$, we define R^n inductively by:

$$R^0 := I_A \quad \text{and} \quad R^{n+1} = R^n \circ R$$

Intuitively, $\langle x, y \rangle$ means that x can "reach" y via n-steps. One can check that $R^m \circ R^n = R^n \circ R^m = R^{n+m}$. For example, for $A = \{a, b, c, d\}$ and relation $R = \{\langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, d\rangle\} \subseteq A \times A$, we have



$R^0 = I_A$ $\quad$ $R^1 = R$ $\quad$ $R^2 = R^4$ $\quad$ $R^3 = R^5$

- Using the above result, we show how to construct transitive closures.

Theorem: Construction of Transitive Closure

Let $R \subseteq A \times A$. Then $t(R) = R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{k=1}^{\infty} R^k$.

Proof: "$\subseteq$" To show that $t(R) \subseteq \bigcup_{k=1}^{\infty} R^k$, since $R \subseteq \bigcup_{k=1}^{\infty} R^k$, it suffices to show that the $\bigcup_{k=1}^{\infty} R^k$ is transitive because $t(R)$ is the smallest transitive relation containing R. To this end, let us consider any $\langle x, y\rangle \in R \cup R^2 \cup R^3 \cup \cdots$ and $\langle y, z\rangle \in R \cup R^2 \cup R^3 \cup \cdots$. This means that $(\exists n)\langle x, y\rangle \in R^n$ and $(\exists m)\langle y, z\rangle \in R^m$, which means that $\langle x, z\rangle \in R^m \circ R^n = R^{m+n} \subseteq R \cup R^2 \cup R^3 \cup \cdots$. Therefore, $\bigcup_{k=1}^{\infty} R^k$ is transitive.

"$\supseteq$" To prove $R \cup R^2 \cup R^3 \cup \cdots \subseteq t(R)$, it suffices to prove $R^n \subseteq t(R)$ for any n. We prove this by induction:

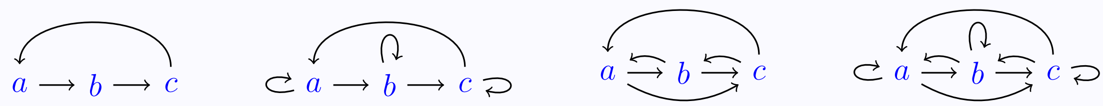
  - For $n = 1$, we have $R \subseteq t(R)$ by definition.
  - Now, we assume that $R^k \subseteq t(R)$ and consider the case of $k + 1$. For any $\langle x, y\rangle \in R^{k+1} = R^k \circ R$, we have that there exists $z \in A$ such that $\langle x, z\rangle \in R \wedge \langle z, y\rangle \in R^k$. By the induction hypothesis, we have $\langle x, z\rangle \in t(R) \wedge \langle z, y\rangle \in t(R)$. However, since $t(R)$ is transitive, we have $\langle x, y\rangle \in t(R)$.

Therefore, we know that $R \cup R^2 \cup R^3 \cup \cdots \subseteq t(R)$.

- The above formula suggests that we may need to compute infinite R^k. However, because R^k will eventually become "periodic" within n-steps, we further have

$$t(R) = \bigcup_{k=1}^{\infty} R^k = R \cup R^2 \cup \cdots \cup R^n$$

In fact, we do not even need to compute until n. As long as we meet a repeated R^k, we can stop. For example, for $A = \{a, b, c\}$ and relation $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, a\rangle\}$, we have



R $\quad$ $r(R)$ $\quad$ $s(R)$ $\quad$ $t(R)$

Multiple Closures

- In some cases, we may want to construct a closure having multiple properties. For example, we may want to find a "reflexive+symmetric+transitive" closure. Then the question is that what is the right order for computing this.

Theorem

① If R is reflexive, then $s(R)$ and $t(R)$ are also reflexive.
② If R is symmetric, then $r(R)$ and $t(R)$ are also symmetric.
③ If R is transitive, then $r(R)$ is also transitive.

Proof of ①: Because R is reflexive, we know that $I_A \subseteq R \subseteq s(R)$ and $I_A \subseteq R \subseteq t(R)$, which means that $s(R)$ and $t(R)$ are also reflexive.

Proof of ②: Because R is symmetric, we have $R^{-1} = R$. For $r(R)$, we have

$$(r(R))^{-1} = (R \cup I_A)^{-1} = R^{-1} \cup I_A^{-1} = R \cup I_A = r(R)$$

For $t(R)$, we have

$$\begin{aligned}(t(R))^{-1} = (R \cup R^2 \cup \cdots)^{-1} =& R^{-1} \cup (R \circ R)^{-1} \cup (R \circ R \circ R)^{-1} \cup \cdots \\ =& R^{-1} \cup (R^{-1} \circ R^{-1}) \cup (R^{-1} \circ R^{-1} \circ R^{-1}) \cup \cdots \\ =& R \cup (R \circ R) \cup (R \circ R \circ R) \cup \cdots \\ =& t(R)\end{aligned}$$

Proof of ③: Because R is transitive, we have $R \circ R \subseteq R$. Then

$$\begin{aligned}r(R) \circ r(R) =& (R \cup I_A) \circ (R \cup I_A) = (R \circ R) \cup (I_A \circ R) \cup (R \circ I_A) \cup (I_A \circ I_A) \\ =& (R \circ R) \cup R \cup I_A \subseteq R \cup I_A = r(R)\end{aligned}$$

- We note that R is transitive does not necessarily imply that $s(R)$ is transitive. To see this, let us consider relation R : $a \longrightarrow b$, which is transitive. Its reflexive closure $s(R) = R \cup R^{-1}$ is $a \rightleftarrows b$, which is not transitive because $\langle a, a \rangle$ and $\langle b, b \rangle$ are missing.

- Base on the above discussion, we have the following results:

Theorem

① $r(s(R)) = s(r(R))$ ② $r(t(R)) = t(r(R))$ ③ $s(t(R)) \subseteq t(s(R))$

- Therefore, the "reflexive+symmetric+transitive" closure of R is actually

$$t(s(r(R))) = t(r(s(R))) = r(t(s(R)))$$

In any case, we should compute the symmetric closure before computing the transitive closure.

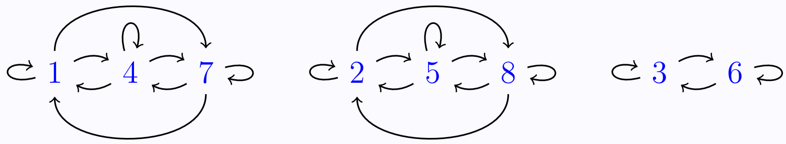
Equivalence Relation

- One of the most widely used relation is "equivalence". By equivalence, in general, we may not require that two objects are exactly the same, but may require that they have the same property of interest. Clearly, an object should be "equal" to itself and if a is equal to b, then b should be equal to a. Also, if a is equal to b and b is equal to c, then a should also be equal to c. This leads to the following definition.

Definition: Equivalence Relation

A relation $R \subseteq A \times A$ is said to be an **equivalence relation (等价关系)** if it is reflexive, symmetric and transitive.

- Let $A = \{1, 2, \ldots, 8\}$. Then $R = \{\langle x, y\rangle : x - y \text{ is divisible by } 3\}$ is an equivalence relation. Its relation graph is shown as follows



Based on the above example, we see that, in an equivalence relation, elements are divided as "groups" such that (i) all elements within each group are related to each other; and (ii) no elements in different groups are related.

Definition: Equivalence Class and Quotient Set

Let $R \subseteq A \times A$ be an equivalence relation. For any $x \in A$, the **equivalence class (等价类)** of x is $[x]_R = \{y : y \in A \wedge xRy\}$. The set of all equivalence classes of A is called the **quotient set (商集)** of A, denoted by $A/R = \{[x]_R : x \in A\}$.

Still, for the above example, we have $[1]_R = [4]_R = [7]_R = \{1, 4, 7\}$, $[2]_R = [5]_R = [8]_R = \{2, 5, 8\}$, $[3]_R = [6]_R = \{3, 6\}$ and $A/R = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6\}\}$.

- Given a non-empty set A, a partition simply divides A into a set of smaller disjoint sets.

Definition: Partition

Let $A \neq \emptyset$ be an non-empty set. We say $\pi \subseteq 2^A$ is a **partition (划分)** of A if:
(i) $\emptyset \notin \pi$; (ii) $\bigcup \pi = A$; and (iii) $(\forall x)(\forall y)((x \in \pi \wedge y \in \pi \wedge x \neq y) \rightarrow x \cap y = \emptyset)$

For equivalence relation $R \subseteq A \times A$, its quotient set A/R is actually a partition of A:

- First, $\emptyset \neq [x]_R \subseteq A$ because $x \in [x]_R$ for any A.
- Also, $\bigcup A/R = A$. For "$\supseteq$", we have $x \in A \Rightarrow x \in [x]_R \in A/R \Rightarrow x \in \bigcup A/R$. For "$\subseteq$", we have $[x]_R \subseteq A \Rightarrow \bigcup\{[x]_R\} \subseteq A$.
- Finally, if xRy, then $[x]_R = [y]_R$, and if $x \not R y$, then $[x]_R \cap [y]_R = \emptyset$.

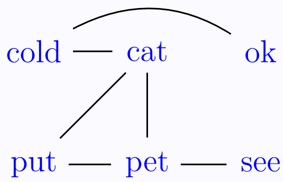
Tolerance Relation

- In some cases, equivalence relation may be too strong because it requires transitivity. For some relations, such as "friends", it needs not be transitive. This leads to the following definition of tolerance relation.

Definition: Tolerance Relation

A relation $R \subseteq A \times A$ is said to be an **tolerance relation (相容关系)** if it is reflexive and symmetric.

- Let $A = \{\text{cold, cat, put, pet, see, ok}\}$. Then $R = \{\langle x, y\rangle : x, y \text{ has a common letter}\}$ is a tolerance relation. Its relation graph is shown as follows



Note that, the above graph takes the following simplification rules. Because it is reflexive, we omit the self-loop at each node. Because it is symmetric, we simply use an undirected line to connect two elements a, b when $\langle a, b\rangle, \langle b, a\rangle \in R$.

- For tolerance relation, unlike equivalence relation, we cannot find equivalence classes that have no overlap. However, we can find the tolerance class.

Definition: Tolerance Class

Let $R \subseteq A \times A$ be a tolerance relation. Then $C \subseteq A$ is called a **tolerance class (相容类)** if

$$(\forall x)(\forall y)((x \in C \wedge y \in C) \to xRy)$$

If there is no tolerance class $C' \subseteq A$ such that $C \subset C'$, then C is called a **maximal tolerance class (最大相容类)**.

Still, for the above example, $\{\text{cold}\}, \{\text{ok}\}, \{\text{put}\}$ are tolerance class but not maximal. $\{\text{cold, cat}\}, \{\text{cat, put, pet}\}, \{\text{ok, cold}\}, \{\text{pet, see}\}$ are maximal tolerance classes.

- For partition, we require that each pair of partition blocks are disjoint. However, for tolerance relation, if we want to contain all elements in A, this may not be possible because some elements have to be used multiple times.

Definition: Cover

Let $A \neq \emptyset$ be an non-empty set. We say $\Omega \subseteq 2^A$ is a **cover (覆盖)** of A if:
(i) $\emptyset \notin \pi$; and (ii) $\bigcup \pi = A$.

Then given tolerance relation R, the set of all maximal tolerance classes is actually a cover. For example, $\Omega = \{\{\text{cold, cat}\}, \{\text{cat, put, pet}\}, \{\text{ok, cold}\}, \{\text{pet, see}\}\}$

Partial Order

- In addition to "equivalence", another important topic is to "compare" each element. Note that, not all elements need to be compared. For example a and b can both be the leader of c but they are not the leader of each other. However, if a is "larger" than b and b is "larger" than c, then a needs to be "larger" than c; otherwise, we will fall into a loop.

Definition: Partial Order

A relation $R \subseteq A \times A$ is said to be a **partial order (偏序)** if it is reflexive, anti-symmetric and transitive. If we associate set A a partial order R, then we call $\langle A, R\rangle$ is **partially-ordered set (偏序集合)** or **poset**.

- Given poset $\langle A, R\rangle$, essentially xRy means that y is "greater than" x. With this understanding, partial order relation R is usually denoted just by $\leq$ directly.

Example

① Let $A = \mathbb{R}$ and $R = \{\langle x, y\rangle : x \leq y\}$, where "$\leq$" is the standard "smaller than or equal to" for real numbers. Then $\langle \mathbb{R}, \leq\rangle$ is a poset. In fact, all elements in $\langle \mathbb{R}, \leq\rangle$ can be compared. However, it is not the general case.

② Let B be a set, $A = 2^B$ be its power set and $R = \{\langle x, y\rangle : x \subseteq y\}$. Then $\langle 2^B, \subseteq\rangle$ is a poset. However, we may have $\{a\} \subseteq \{a, b\}$, $\{a\} \subseteq \{a, c\}$ but $\{a, b\}$ and $\{a, c\}$ cannot be compared. This is why we call it partial order.

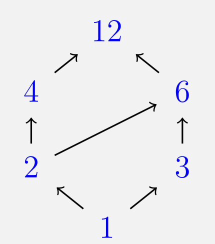
- Based on the above example, we see that not all elements in $\langle A, \leq\rangle$ can be compared. For any $x, y \in A$, we say x and y are **comparable (可比的)** if $(x \leq y) \vee (y \leq x)$. Furthermore, because partial order is transitive, by knowing $a \leq b$ and $b \leq c$, we know automatically that $a \leq c$. Therefore, it suffices to remember the "closest" two comparable elements. Formally, we say y **covers (盖住)** x if

$$(x \leq y) \wedge (x \neq y) \wedge \neg(\exists z)(z \in A \wedge x \leq z \wedge z \leq y \wedge z \neq x \wedge z \neq y)$$

We define $\mathsf{cov}_R(A) = \{\langle x, y\rangle \in R : y \text{ covers } x\}$. Therefore, $\mathsf{cov}_R(A)$ contains all information of $\langle A, R\rangle$. The relation graph of $\mathsf{cov}_R(A)$ is called the **Hasse Diagram (哈斯图)**, which is sufficient enough to represent a poset $\langle A, R\rangle$.

Example: Hasse Diagram

Let $A = \mathbb{N} \setminus \{0\}$ and $R = \{\langle x, y\rangle : x \text{ divides } y\}$. One can show that $\langle A, R\rangle$ is a poset. For example, consider $A = \{1, 2, 3, 4, 6, 12\}$ and $R = \{\langle x, y\rangle : x \text{ divides } y\}$, we have $\mathsf{cov}_R(A) = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 4\rangle, \langle 2, 6\rangle, \langle 3, 6\rangle, \langle 4, 12\rangle, \langle 6, 12\rangle\}$. Its Hasse graph is shown on the right.



Upper Bound and Lower Bound

- Given a poset $\langle A, \leq \rangle$, we may want to find a "largest" element. There are two interpretations: (i) a is larger than any other element; (ii) no element is larger than a. Note that this two interpretations are different because we only have partial order. The first case is actually called greatest (最大), while the second case is actually called maximal (极大).

Definition: Least/Greatest

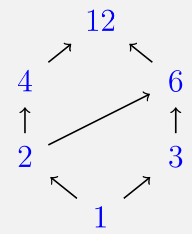
Let $\langle A, \leq \rangle$ be a poset and $B \subseteq A$ be a subset. For any $x \subseteq B$, we say that x is

- the **least element (最小元)** of B if $(\forall y)(y \in B \to x \leq y)$
- the **greatest element (最大元)** of B if $(\forall y)(y \in B \to y \leq x)$
- a **minimal element (极小元)** of B if $(\forall y)(y \in B \land y \leq x \to x = y)$
- a **maximal element (极大元)** of B if $(\forall y)(y \in B \land x \leq y \to x = y)$

- We note that if the least/greatest element exists, then it is unique. However, it may not exist even when A is finite. On the other hand, maximal/minimal element always exists for finite A but may not be unique due to incomparable maximal/minimal.

Let us consider poset $\langle A, \leq \rangle$ shown on the right.

- If we consider $B = A$, then 12 is the greatest element (also maximal) and 1 is the smallest element (also minimal).
- If we consider $B = \{2, 3, 4, 6\}$, then there is no greatest/least element. There are two maximal elements 4 and 6, and two minimal elements 2 and 3.

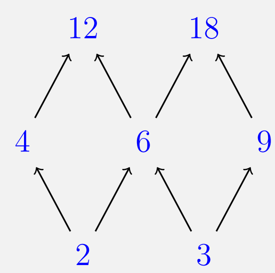


- Given poset $\langle A, \leq \rangle$ and subset $B \subseteq A$, we say that $x \in A$ is
  - the **upper bound (上界)** of B if $(\forall y)(y \in B \to y \leq x)$
  - the **lower bound (下界)** of B if $(\forall y)(y \in B \to x \leq y)$

  Note that the upper/lower bounds of B do not need in B; it can be in $A \setminus B$. Also, upper/lower bounds is not unique because they can be very rough. We further define (i) the **least upper bound (最小上界/上确界)** of B as the least element of the set of all upper bounds of B; and (ii) the **greatest lower bound (最大下界/下确界)** of B is the greatest element of the set of all lower bounds of B.

Let us consider poset $\langle A, \leq \rangle$ shown on the right.

- For $B_1 = \{2, 4\}$, upper bound$= 4, 12$ with l.u.b.$= 4$, lower bound=g.l.b$= 2$.
- For $B_2 = \{4, 6, 9\}$, there is no upper/lower bound.
- For $B_3 = \{2, 3\}$, upper bound$= 6, 12, 18$ with l.u.b.$= 6$. There is no lower bound.



Total Order and Well Order

- Note that in partial order, some elements may not be comparable. For poset $\langle A, \leq \rangle$, we further call it
  - a **totally-ordered set (全序集)** if any two elements in A are comparable;
  - a **well-ordered set (良序集)** if any non-empty subset of A has a least element.
- We note that a well-ordered set is always totally-ordered. To see this, suppose $\langle A, \leq \rangle$ is well-ordered. Then for any $x, y \in A$, subset $\{x, y\} \subseteq A$ has a least element, which is either x or y. Therefore, we have either $x \leq y$ or $y \leq x$, i.e., x and y are comparable. So $\langle A, \leq \rangle$ is totally-ordered.
- However, a totally-ordered set may not be well-ordered. As a counter-example, let us consider $\langle \mathbb{R}, \leq \rangle$, which is totally-ordered. However, for subset $(0, 1] = \{x : x \in \mathbb{R} \wedge 0 \leq x \leq 1 \wedge x \neq 0\}$, it has no least element. Therefore, it is not well-ordered.