6 Stopping Times & Applications of Martingale Theory

Stopping Times

- In games of chance, you usually have the option to quit at any time. For example, you can fix in advance that you will quit after k rounds. Or you can decide to quit either when you bankrupt or when you win enough many. Therefore, you need a condition to trigger when you quit, which is called the **stopping time**.
- Note that a stopping time is not a fixed number because it depends on the specific realization of your process. Therefore, a stopping time is a random variable $\tau : \Omega \to \{1, 2, \dots\} \cup \{\infty\}$.
- Furthermore, after the outcome at instant n, the information you have is $\mathcal{F}_n$. Therefore, your information must be sufficient enough to support your decision.

Definition: Stopping Times

A random variable $\tau : \Omega \to \{1, 2, \dots\} \cup \{\infty\}$ is said to be a *stopping time* w.r.t. a filtration $\{\mathcal{F}_n\}$ if

$$\forall n = 1, 2, \cdots : \{\tau = n\} = \{\omega \in \Omega : \tau(\omega) = n\} \in \mathcal{F}_n$$

- A naive stopping time is $\forall \omega \in \Omega : \tau(\omega) = k$ that fixes the quit time in advance. This is indeed a stopping time because $\{\tau = n\} = \emptyset$ when $n \neq k$ and $\{\tau = n\} = \Omega$ when $n = k$.
- One of the most commonly used stopping time is the *time of first entry*. Formally, let $\{X_n\}$ be a process adapted to filtration $\{\mathcal{F}_n\}$ and let $B \in \mathcal{B}(\mathbb{R})$ be a Borel set. Then the following random variable $\tau : \Omega \to \{1, 2, \dots\} \cup \{\infty\}$ is a stopping time

$$\tau = \min\{n : X_n \in B\}$$

To see why it is a stopping time, for any $n = 1, 2, \dots,$ we write

$$\{\tau = n\} = \{X_1 \notin B\} \cap \{X_2 \notin B\} \cap \dots \cap \{X_{n-1} \notin B\} \cap \{X_n \in B\}$$

Because B is a Borel set, we have each of the sets on the right-hand side above belongs to $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Therefore, their intersection also belongs to $\mathcal{F}_n$.

- **Remark: An Equivalent Definition**
In some textbook, stopping times are defined by $\{\tau \leq n\} \in \mathcal{F}_n$ rather than $\{\tau = n\} \in \mathcal{F}_n$. However, these two conditions are equivalent:
  - If $\forall n = 1, 2, \dots : \{\tau \leq n\} \in \mathcal{F}_n$, then $\forall n = 1, 2, \dots : \{\tau \leq n-1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$. So

$$\forall n = 1, 2, \dots : \{\tau = n\} = \{\tau \leq n\} \setminus \{\tau \leq n-1\} \in \mathcal{F}_n$$

  - If $\forall n = 1, 2, \dots : \{\tau = n\} \in \mathcal{F}_n$, then $\forall k = 1, 2, \dots, n : \{\tau = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$. So

$$\forall n = 1, 2, \dots : \{\tau \leq n\} = \{\tau = 1\} \cup \dots \cup \{\tau = n\} \in \mathcal{F}_n$$

Stopped Processes

- Given a process $\{X_n\}$ adapted to filtration $\{\mathcal{F}_n\}$ and a stopping time τ. Essentially, τ truncates the process at the instant of $\tau(\omega)$. That is, after you quit the game, your total capital remains unchanged.
- For any numbers a and b, we denote $a \wedge b = \min\{a, b\}$. Then we call $\{X_{\tau\wedge n}\}$ **the process stopped at** τ, which is also denoted by X_n^τ. Specifically, we have

$$\forall \omega \in \Omega : X_n^\tau(\omega) = X_{\tau(\omega)\wedge n}(\omega)$$

- The stopped process $\{X_{\tau\wedge n}\}$ is also adapted to $\{\mathcal{F}_n\}$. To see this, for any Borel set $B \in \mathcal{B}(\mathbb{R})$, we can write

$$\{X_{\tau\wedge n} \in B\} = \{X_n \in B, \tau > n\} \cup \bigcup_{k=1}^{n} \{X_k \in B, \tau = k\}$$

  where $\{X_n \in B, \tau > n\} = \{X_n \in B\} \cap \{\tau > n\} \in \mathcal{F}_n$ and for each $k = 1, \ldots, n$, we have

$$\{X_k \in B, \tau = k\} = \{X_k \in B\} \cap \{\tau = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$$

  Therefore, we have $X_{\tau\wedge n} \in \mathcal{F}_n$

Elementary Stopping Theorem

- In fact, we can consider a stopping time $\tau : \Omega \to \{1, 2, \ldots\} \cup \{\infty\}$ as a gambling strategy (previsible process) $\{C_n\}$ defined by

$$C_n = \begin{cases} 1 & \text{if } \tau \geq n \\ 0 & \text{if } \tau < n \end{cases}$$

  To see why C_n is $\mathcal{F}_{n-1}$-measurable, for any $B \in \mathcal{B}(\mathbb{R})$, there are four cases for $\{C_n \in B\}$:

  - If $0 \notin B$ and $1 \notin B$, then $\{C_n \in \emptyset\} = \emptyset \in \mathcal{F}_{n-1}$
  - If $0 \in B$ and $1 \in B$, then $\{C_n \in \{0, 1\}\} = \Omega \in \mathcal{F}_{n-1}$
  - If $0 \in B$ and $1 \notin B$, then $\{C_n = 0\} = \{\tau < n\} = \{\tau \leq n - 1\} \in \mathcal{F}_{n-1}$
  - If $0 \notin B$ and $1 \in B$, then $\{C_n = 1\} = \{\tau \geq n\} = \{\tau > n - 1\} \in \mathcal{F}_{n-1}$

  Therefore, the $X_{\tau\wedge n}$ is the martingale transform

$$X_{\tau\wedge n} = (C \bullet X)_n = C_1(X_1 - X_0) + \cdots + C_n(X_n - X_{n-1})$$

  This gives use the following theorem

Elementary Stopping Theorem

Let τ be a stopping time. If X_n is a (super or sub) martingale, then so is $X_{\tau\wedge n}$.

Doob's Optional Stopping Theorem

- When $\{X_n\}$ is a martingale, we know that $E(X_n) = E(X_1)$. Furthermore, the elementary stopping theorem just makes sure that $E(X_{\tau\wedge n}) = E(X_1)$ for any stopping time τ. **However, it does not mean that $\mathbf{E(X_\tau) = E(X_1)}$!**

- For example, when we play the martingale strategy, we have $E(X_\tau) = 1 \neq E(X_1) = 0$, although $E(X_{\tau\wedge n}) = E(X_1) = 0$ (check this by yourself). The key issue is that, the gambler's expected loss just before the ultimate win is infinite, i.e.,

$$E(X_{\tau-1}) = \sum_{n=1}^{\infty} X_{n-1}P(\tau = n)) = \sum_{n=1}^{\infty} \frac{-1-2-\cdots-2^{n-2}}{2^n} = -\sum_{n=1}^{\infty} \frac{2^{n-1}-1}{2^n} = -\infty$$

- Another example is the random walk with absorbing barriers. Specifically, let $Y_1, Y_2, \ldots$ be i.i.d. with $P(Y_i = 1) = P(Y_i = -1) = 0.5$ and let $X_n = Y_1 + \cdots + Y_n$. We have shown that $\{X_n\}$ is a martingale. Let $\tau = \min\{n : X_n = 1\}$. Then we have $E(X_{\tau\wedge n}) = E(X_1) = 0$. However, since $P(\tau < \infty) = 1$, we have $E(X_\tau) = 1 \neq 0 = E(X_1)$.

- Therefore, it will be very useful to investigate, under what condition, we further have $E(X_\tau) = E(X_1)$. This is given by the following theorem.

Doob's Optional Stopping Theorem

Let X_n be a super-martingale and τ be a stopping time. Then X_τ is integrable and

$$E(X_\tau) \leq E(X_1)$$

if **one of** the following conditions hold:

1. τ is bounded, i.e., $\exists N \in \mathbb{N}, \forall \omega \in \Omega : \tau(\omega) < N$;
2. X is bounded, i.e., $\exists K \in \mathbb{R}, \forall \omega \in \Omega, \forall n \in \mathbb{N} : |X_n(\omega)| < K$, and τ is almost surely finite;
3. $E(\tau) < \infty$ and $\exists K \in \mathbb{R}, \forall n \in \mathbb{N}, \omega \in \Omega : |X_n(\omega) - X_{n-1}(\omega)| \leq K$.

If any of the above conditions holds and X is a martingale, then

$$E(X_\tau) = E(X_1)$$

Proof: Since $\{X_n\}$ is a super-martingale, we know that $\{X_{\tau\wedge n}\}$ is also a super-martingale, which is integrable and

$$E(X_{\tau\wedge n} - X_1) \leq 0$$

(1) is straightforward by choosing $n = N$.
(2) follows from the bounded convergence theorem, which says that, if $X_n \to X$ almost surely and for some $K, |X_n(\omega)| \leq K, \forall n, \omega$, then $E(|X_n - X|) \to 0$.
(3) follows from the observation that

$$|X_{\tau\wedge n} - X_1| = |\sum_{k=1}^{\tau\wedge n}(X_k - X_{k-1})| \leq K\tau$$

Applications of Martingales: The First Run of Three Sixes

- A fair die is thrown independently at each time instant. A gambler wins a fixed amount of money as soon as the first run of *three consecutive sixes* appears.

  What is the expected number of the throws of the dice until the gambler wins for the first time?

  Formally, let $X_1, X_2, \ldots$ be the outcomes of the throws, which are i.i.d. with $P(X_i = k) = 1/6, k = 1, \ldots, 6$. Write $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and let τ be the first time three consecutive sixes appears. Clearly, τ is a stopping time and we are looking for $E(\tau)$.

  First, we can roughly estimate $E(\tau) < \infty$. To have $\tau = k$, we need to have one number different from six in every tuple $(X_{3m+1}, X_{3m+2}, X_{3m+3}), 3m + 3 < k$. Hence we have

  $$P(\tau = k) \leq \left(1 - (1/6)^3\right)^{\frac{k-3}{3}}$$

  This means that $E(\tau) = \sum_{k=1}^{\infty} kP(\tau = k)$ converges.

  Let us consider the following thought experiment. Suppose that just before each time n, a gambler appears on the scene and bets ¥1 that the nth throw will show six. If he loses, he leaves; otherwise he receives ¥6 and uses all these money to bet that the $(n + 1)$th throw shows six. Again, if he loses, he leaves; otherwise he bet ¥36 on a six in the third throw. Since the game is fair, at any time $n \geq 3$, the expected winnings should be equal to the total money spent by the gamblers up to time n. As τ is a stopping time satisfying the condition of optional stopping theorem, this should hold at time T as well, hence

  $$E(\tau) = 6 + 6^2 + 6^3 = 258$$

  Indeed, $E(\tau)$ is the expected money spent by the gamblers, and at time τ the last gambler has won ¥6, the one before has won ¥36 and the one before that has won ¥216. All other gambler have lost their stakes.

  To be more specific, let S_n be the total stakes of all players at time n, i.e., $S_n = 1 + 6 + \cdots + 6^k$ if we are in a run of k sixes at time n, and let $M_n = S_n - n$, where $M_0 = 1$. Then $\{M_n\}$ is a martingale because

  $$E(M_{n+1} \mid \mathcal{F}_n) = \frac{5}{6}(1 - (n + 1)) + \frac{1}{6}(6S_n + 1 - (n + 1)) = S_n - n = M_n$$

  For the stopped martingale $\{M_{\tau \wedge n}\}$, since $E(\tau) < \infty$ and $|M_n - M_{n-1}| \leq 260$, by applying the Doob's Optional Stopping Theorem, we have

  $$1 = E(M_0) = E(M_\tau) = E(S_\tau) - E(\tau) = 1 + 6 + 6^2 + 6^3 - E(\tau)$$

Applications of Martingales: The Gambler's Ruin Problem

- Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $P(X_i = 1) = p$ and $P(X_i = -1) = q = 1 - p$ such that $p > 0.5$. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and consider $0 < a < b$. Let $S_n = a + X_1 + \cdots + X_n$ be the capital of a gambler starting with a capital a in a favourable game. Suppose that the gambler stops when he is bankrupt or when his capital reaches b. The the stopping time is

$$\tau = \min\{n : S_n = 0 \text{ or } S_n = b\}$$

We want to find (i) the probability of ruin $P(S_\tau = 0)$, (ii) the expected time of the game $E(\tau)$, and (iii) the expected capital when the game ends $E(S_\tau)$.

One can easily argue that $E(\tau) < \infty$ (check by yourself). Then we choose

$$M_n = \left(\frac{q}{p}\right)^{S_n} \text{ and } N_n = S_n - n(p - q)$$

Note that processes $\{M_n\}$ and $\{N_n\}$ are both martingales. For process $\{M_n\}$, we have

$$E(M_{n+1} \mid \mathcal{F}_n) = p\left(\frac{q}{p}\right)^{S_n+1} + q\left(\frac{q}{p}\right)^{S_n-1} = \left(\frac{q}{p}\right)^{S_n}(q + p) = M_n$$

For process $\{N_n\}$, we have

$$E(N_{n+1} \mid \mathcal{F}_n) = E(S_{n+1} \mid \mathcal{F}_n) - (n+1)(p-q) = S_n + p - q - (n+1)(p-q) = N_n$$

For process $\{M_n\}$, since $|M_n - M_{n-1}| \leq 1$, we have

$$\left(\frac{q}{p}\right)^{a} = E(M_0) = E(M_\tau) = P(S_\tau = 0) + \left(\frac{q}{p}\right)^{b} P(S_\tau = b)$$

Since $P(S_\tau = 0) = 1 - P(S_\tau = b)$, we obtain

$$P(S_\tau = 0) = \frac{\left(\frac{q}{p}\right)^{a} - \left(\frac{q}{p}\right)^{b}}{1 - \left(\frac{q}{p}\right)^{b}}$$

Also, we can obtain

$$E(S_\tau) = bP(S_\tau = b) = b \times \frac{1 - \left(\frac{q}{p}\right)^{a}}{1 - \left(\frac{q}{p}\right)^{b}}$$

For process $\{N_n\}$, since $|N_n - N_{n-1}| \leq 1 + p - q$, we have

$$a = E(N_0) = E(N_\tau) = E(S_\tau - \tau(p-q)) = E(S_\tau) - E(\tau)(p-q)$$

This gives us

$$E(\tau) = \frac{1}{p-q}(E(S_\tau) - a) = \frac{b}{p-q}\frac{1 - \left(\frac{q}{p}\right)^{a}}{1 - \left(\frac{q}{p}\right)^{b}} - \frac{a}{p-q}$$

Applications of Martingales: The Secretary Problem

- N candidates present themselves for a job interview. The ith candidate's suitability for the job is X_i, which are independent and uniformly distributed on $[0, 1]$. The boss interviews each in turn and can determine the value of X_i perfectly. He must immediately decide whether to accept or reject the candidate, no recall of rejected candidates is possible.

  The problem is that the boss has to find a stopping time τ which maximizes $E(X_\tau)$. We now use martingale theory to solve this problem.

  Claim: The stopping time $\tau^* = \min\{n : X_n > \alpha_n\}$, where

$$\alpha_N = 0 \quad \text{and} \quad \alpha_{n-1} = \frac{1}{2} + \frac{\alpha_n^2}{2}$$

  solves the problem.

  Step 1: For any $0 \le \alpha \le 1$, we have $E(X_n \vee \alpha) = \frac{1}{2} + \frac{\alpha^2}{2}$. This can be seen

$$E(X_n \vee \alpha) = \int_0^1 X_n \vee \alpha dx = \int_0^\alpha \alpha dx + \int_\alpha^1 x dx = \alpha^2 + \frac{1}{2} - \frac{\alpha^2}{2} = \frac{1}{2} + \frac{\alpha^2}{2}$$

  Step 2: For any stopping time τ, we define a new process $\{Y_n\}$

$$Y_0 = \alpha_0 \quad \text{and} \quad Y_n = (X_{\tau\wedge n}) \vee \alpha_n$$

  Show by yourself that $\{Y_n\}$ is a super-martingale.

  Step 3: We show that $\tau = \tau^*$ is actually a martingale still by the following two cases.

  Step 4: We show that for any stopping time τ, we have $E(X_\tau) \le E(X_{\tau^*})$. Since the stopping time is bounded, we can apply Doob's theorem to obtain

$$E(X_\tau) \le E(X_\tau \vee \alpha_\tau) = E(Y_\tau) \le E(Y_0) = \alpha_0$$

  Furthermore, for the specific choice τ^*, we have

$$E(X_{\tau^*}) = E(X_{\tau^*} \vee \alpha_\tau) = E(Y_{\tau^*}) = E(Y_0) = \alpha_0$$

  This completes the proof.

Applications of Martingales: The Second Hearts Problem

- In a deck of 52 cards, well-shuffled, we turn the cards from the top until the first $\heartsuit$ appears. If we turn one more card, what is the probability that this card shows $\heartsuit$ again?

  Let X_n be the proportion of $\heartsuit$ remaining in the deck after the nth card is turned. Let Y_n be the indicator of the event that the nth card is $\heartsuit$, and let $\mathcal{F}_n = \sigma(Y_0, \dots, Y_n) = \sigma(X_0, \dots, X_n)$. We claim that $\{X_n : 0 \le n \le 51\}$ is a martingale.

$$E(X_n \mid \mathcal{F}_{n-1}) = \frac{(53-n)X_{n-1} - 1}{52-n} X_{n-1} + \frac{(53-n)X_{n-1}}{52-n}(1 - X_{n-1}) = X_{n-1}$$

  Now we let $\tau = \min\{n : Y_n = 1\}$ be the first time $\heartsuit$ appears. Note that, given X_τ, the probability that the $\tau+1$st card is again hearts is X_τ. Hence the unconditional probability that the $\tau+1$st card is again hearts is $E(X_\tau)$. As τ is a bounded stopping, we can apply the Doob's optional stopping theorem, which gives

$$E(X_\tau) = E(X_0) = 0.25$$